Vectors and the Dot Product

1. Are the following better described by vectors or scalars?

   (a) The cost of a Super Bowl ticket.

   (b) The wind at a particular point outside.

   (c) The number of students at Harvard.

   (d) The velocity of a car.

   (e) The speed of a car.

2. Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?

Definition. The dot product $\vec{v} \cdot \vec{w}$ of two vectors $\vec{v}$ and $\vec{w}$ is defined as follows.

- If $\vec{v}$ and $\vec{w}$ are two-dimensional vectors, say $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2$.
- If $\vec{v}$ and $\vec{w}$ are three-dimensional vectors, say $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2 + v_3 w_3$.

It is not possible to dot a two-dimensional vector with a three-dimensional vector!

3. (a) What is $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$?

   (b) What is $\langle 1, 2, 3 \rangle \cdot \langle 4, -5, 6 \rangle$?

Here are some basic algebraic properties of the dot product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors of the same dimension and c is a scalar, then

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
3. $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (c\vec{w})$.

4. True or false: if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors of the same dimension, then $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.

5. What is the relationship between $\vec{v} \cdot \vec{v}$ and $|\vec{v}|$?

6. Find the angle between $\langle 1, 2, 1 \rangle$ and $\langle 1, -1, 1 \rangle$.

7. Find the vector projection of $\langle 0, 0, 1 \rangle$ onto $\langle 1, 2, 3 \rangle$.

8. True or false: If $\vec{v}$ and $\vec{w}$ are parallel, then $|\vec{v} - \vec{w}| = |\vec{v}| - |\vec{w}|$.

9. If $\vec{v}$ and $\vec{w}$ are vectors with the property that $|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$, which of the following must be true?

   (a) $\vec{v} = \vec{w}$.

   (b) $\vec{v} = \vec{0}$.

   (c) $\vec{v}$ is orthogonal to $\vec{w}$.

   (d) $\vec{v}$ is parallel to $\vec{w}$.

Cross Product and Triple Product

Algebraic definition of the cross product. If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then we define $\vec{v} \times \vec{w}$ to be $\langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1 \rangle$.

There is a handy way of remembering this definition: the cross product $\vec{v} \times \vec{w}$ is equal to the determinant

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k}$$

Note: The cross product is only defined for three-dimensional vectors.

1. For this problem, let $\vec{v} = \langle 1, 2, 1 \rangle$ and $\vec{w} = \langle 0, -1, 3 \rangle$.

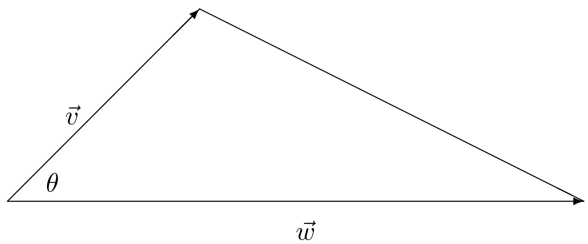
   (a) Compute $\vec{v} \times \vec{w}$.

   (b) Compute $\vec{w} \times \vec{v}$.

   (c) Let $\vec{u} = \vec{v} \times \vec{w}$, the vector you found in (a). What is the angle between $\vec{u}$ and $\vec{v}$? $\vec{u}$ and $\vec{w}$?

2. In general, what is the relationship between $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$?

3. Any two vectors $\vec{v}$ and $\vec{w}$ which are not parallel determine a triangle, as shown. What is the relationship between the area of the triangle and $\vec{v} \times \vec{w}$?



4. If $\vec{v}$ and $\vec{w}$ are parallel, what is $\vec{v} \times \vec{w}$?

5. If the scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ is equal to 0, what can you say about the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$?

6. Find an equation for the plane which passes through the points $(1, 0, 1)$, $(0, 2, 0)$, and $(2, 1, 0)$.

7. True or false: If $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$.

8. True or false: If $\vec{v} \times \vec{w} = \vec{0}$ and $\vec{v} \cdot \vec{w} = 0$, then at least one of $\vec{v}$ and $\vec{w}$ must be $\vec{0}$.

Vectors and the Dot Product

1. Are the following better described by vectors or scalars?

   (a) *The cost of a Super Bowl ticket.*

   Solution. Scalar — the cost is just a number.

   (b) *The wind at a particular point outside.*

   Solution. Vector — the wind has both a speed and a direction.

   (c) *The number of students at Harvard.*

   Solution. Scalar.

   (d) *The velocity of a car.*

   Solution. Vector. The velocity is defined to be both the speed of the car (how fast it's going) and the direction it's going.

   (e) *The speed of a car.*

   Solution. Scalar. The speed refers only to how fast the car is going; it is the magnitude of the velocity vector.

2. *Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?*

   Solution. The force Bert is applying can be described by the vector $\langle 0, 30\rangle$, while the force Ernie is applying is $\langle 40, 0\rangle$. We know that the resultant force can be obtained simply by summing the individual force vectors, so the resultant force is $\boxed{\langle 40, 30\rangle}$.

3. (a) *What is* $\langle 1, 2\rangle \cdot \langle 3, 4\rangle$*?*

   Solution. $1 \cdot 3 + 2 \cdot 4 = \boxed{11}$.

   (b) *What is* $\langle 1, 2, 3\rangle \cdot \langle 4, -5, 6\rangle$*?*

   Solution. $1 \cdot 4 + 2 \cdot -5 + 3 \cdot 6 = \boxed{12}$.

4. *True or false: if* $\vec{u}$, $\vec{v}$, *and* $\vec{w}$ *are vectors of the same dimension, then* $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.

   Solution. Completely false. In fact, the statement doesn't even make sense! $\vec{v} \cdot \vec{w}$ is a scalar, and we can't dot a vector with a scalar, so $\vec{u} \cdot (\vec{v} \cdot \vec{w})$ is meaningless.

5. *What is the relationship between* $\vec{v} \cdot \vec{v}$ *and* $|\vec{v}|$*?*

   Solution. $\vec{v} \cdot \vec{v}$ is equal to $|\vec{v}|^2$. Again, this is easy to see from the component definition. For a two-dimensional vector $\vec{v} = \langle v_1, v_2\rangle$, $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 = |\vec{v}|^2$. For a three-dimensional vector $\vec{v} = \langle v_1, v_2, v_3\rangle$, $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2 = |\vec{v}|^2$.

6. *Find the angle between* $\langle 1, 2, 1\rangle$ *and* $\langle 1, -1, 1\rangle$.

Solution. Let $\vec{v} = \langle 1, 2, 1 \rangle$ and $\vec{w} = \langle 1, -1, 1 \rangle$, and let θ be the angle between $\vec{v}$ and $\vec{w}$. Then, we know that $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$. We calculate that $\vec{v} \cdot \vec{w} = 1 \cdot 1 + 2 \cdot -1 + 1 \cdot 1 = 0$, so $0 = |\vec{v}||\vec{w}| \cos\theta$. Since the lengths $|\vec{v}|$ and $|\vec{w}|$ are both positive, $\cos\theta = 0$, so $\theta = \frac{\pi}{2}$.

7. *Find the vector projection of* $\langle 0, 0, 1 \rangle$ *onto* $\langle 1, 2, 3 \rangle$.

   Solution. Let $\vec{v} = \langle 0, 0, 1 \rangle$ and $\vec{w} = \langle 1, 2, 3 \rangle$. We saw in class that the projection of $\vec{v}$ onto $\vec{w}$ is $\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$. In this case, $\vec{v} \cdot \vec{w} = 3$ and $\vec{w} \cdot \vec{w} = 1^2 + 2^2 + 3^2 = 14$, so the projection is $\frac{3}{14}\langle 1, 2, 3 \rangle = \left\langle \frac{3}{14}, \frac{6}{14}, \frac{9}{14} \right\rangle$.

8. *True or false: If* $\vec{v}$ *and* $\vec{w}$ *are parallel, then* $|\vec{v} - \vec{w}| = |\vec{v}| - |\vec{w}|$.

   Solution. False. For example, let $\vec{v} = \langle 1, 0, 0 \rangle$ and $\vec{w} = -\langle 1, 0, 0 \rangle$. Then, $\vec{v} - \vec{w} = \langle 2, 0, 0 \rangle$, which has length 2. On the other hand, $\vec{v}$ and $\vec{w}$ both have length 1, so $|\vec{v}| - |\vec{w}| = 0$.

9. *If* $\vec{v}$ *and* $\vec{w}$ *are vectors with the property that* $|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$, *which of the following must be true?*

   (a) $\vec{v} = \vec{w}$.

   (b) $\vec{v} = \vec{0}$.

   (c) $\vec{v}$ *is orthogonal to* $\vec{w}$.

   (d) $\vec{v}$ *is parallel to* $\vec{w}$.

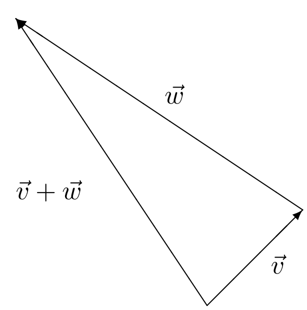
   Solution. (c).

   We can rewrite the equation $|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$ using dot products:

$$\begin{aligned} (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) &= \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} \\ \cancel{\vec{v} \cdot \vec{v}} + 2\vec{v} \cdot \vec{w} + \cancel{\vec{w} \cdot \vec{w}} &= \cancel{\vec{v} \cdot \vec{v}} + \cancel{\vec{w} \cdot \vec{w}} \\ 2\vec{v} \cdot \vec{w} &= 0 \\ \vec{v} \cdot \vec{w} &= 0 \end{aligned}$$

   This is, of course, exactly what it means for $\vec{v}$ to be orthogonal to $\vec{w}$.

   You could also think about this problem geometrically. If $\vec{v}$ and $\vec{w}$ are not parallel, then $\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$ form a triangle:



   The equation $|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$ says that the sides of the triangle satisfy the Pythagorean Theorem, so the triangle must be a right triangle with $\vec{v} + \vec{w}$ as the hypotenuse and $\vec{v}$ and $\vec{w}$ as the two legs. In other words, $\vec{v}$ and $\vec{w}$ must be orthogonal.

Cross Product and Triple Product

1. For this problem, let $\vec{v} = \langle 1, 2, 1 \rangle$ and $\vec{w} = \langle 0, -1, 3 \rangle$.

   (a) *Compute* $\vec{v} \times \vec{w}$.

   Solution. $\langle 7, -3, -1 \rangle$.

   (b) *Compute* $\vec{w} \times \vec{v}$.

   Solution. $\langle -7, 3, 1 \rangle$.

   (c) *Let* $\vec{u} = \vec{v} \times \vec{w}$, *the vector you found in (a). What is the angle between* $\vec{u}$ *and* $\vec{v}$? $\vec{u}$ *and* $\vec{w}$?

   Solution. To find the angle between two vectors, we use the dot product.

   $\vec{u} \cdot \vec{v} = \langle 7, -3, -1 \rangle \cdot \langle 1, 2, 1 \rangle = (7)(1) + (-3)(2) + (-1)(1) = 0$, so $\vec{u}$ is orthogonal to $\vec{v}$.

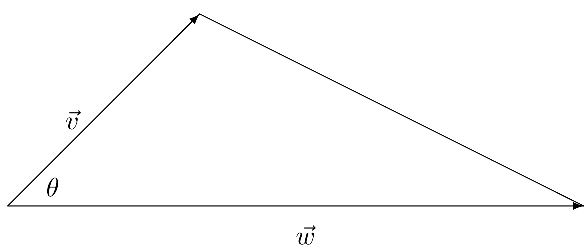
   $\vec{u} \cdot \vec{w} = \langle 7, -3, -1 \rangle \cdot \langle 0, -1, 3 \rangle = (7)(0) + (-3)(-1) + (-1)(3) = 0$, so $\vec{u}$ is also orthogonal to $\vec{w}$.

2. *In general, what is the relationship between* $\vec{v} \times \vec{w}$ *and* $\vec{w} \times \vec{v}$?

   Solution. From the definition of $\vec{v} \times \vec{w}$, you can compute directly that $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

   Here's another way to get the same conclusion. Let θ be the angle between $\vec{v}$ and $\vec{w}$. Then, we know that the length of $\vec{v} \times \vec{w}$ is $|\vec{v}||\vec{w}| \sin\theta$, and the length of $\vec{w} \times \vec{v}$ is $|\vec{w}||\vec{v}| \sin\theta$. So, $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ have the same length. Also, $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ are both orthogonal to both $\vec{v}$ and $\vec{w}$, so they are either the same vector or negatives of each other. The right-hand rule tells us they must point in opposite directions, so $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

3. *Any two vectors* $\vec{v}$ *and* $\vec{w}$ *which are not parallel determine a triangle, as shown. What is the relationship between the area of the triangle and* $\vec{v} \times \vec{w}$?



   Solution. The base of the triangle has length $|\vec{w}|$, and the height of the triangle is $|\vec{v}| \sin\theta$, so the area of the triangle is $\frac{1}{2}|\vec{v}||\vec{w}| \sin\theta$, which is equal to $\frac{1}{2}|\vec{v} \times \vec{w}|$.

   (Note that it is NOT correct to say that the area of the triangle is half of the **vector** $\vec{v} \times \vec{w}$; after all, area is a scalar, not a vector. Rather, the area of the triangle is half of the **length** of the vector $\vec{v} \times \vec{w}$).

4. *If* $\vec{v}$ *and* $\vec{w}$ *are parallel, what is* $\vec{v} \times \vec{w}$?

   Solution. If $\vec{v}$ and $\vec{w}$ are parallel, then the angle θ between them is either 0 or π. In either case, $\sin\theta = 0$, so $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta = 0$. This means that $\vec{v} \times \vec{w}$ must be the zero vector $\vec{0}$.

5. *If the scalar triple product* $\vec{u} \cdot (\vec{v} \times \vec{w})$ *is equal to* 0, *what can you say about the vectors* $\vec{u}$, $\vec{v}$, *and* $\vec{w}$?

Solution. The fact that $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ means that $\vec{u}$ is orthogonal to $\vec{v} \times \vec{w}$. We also know that $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$, so this means that $\vec{u}, \vec{v}, \vec{w}$ are *all* orthogonal to the vector $\vec{v} \times \vec{w}$. In particular, if we stick the tails of $\vec{u}$, $\vec{v}$, and $\vec{w}$ at the same point, then $\vec{u}$, $\vec{v}$, and $\vec{w}$ all lie in the same plane. (We say that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are coplanar.)

Note: Some students pointed out a special case: $\vec{u} \cdot (\vec{v} \times \vec{w})$ is equal to 0 if $\vec{v}$ and $\vec{w}$ are parallel (since then $\vec{v} \times \vec{w} = \vec{0}$, by #4). In this case, $\vec{u}$, $\vec{v}$, and $\vec{w}$ are still coplanar. To visualize this, imagine the plane that contains $\vec{u}$ and $\vec{v}$: $\vec{w}$ will automatically be in this plane because it's parallel to $\vec{v}$.

6. *Find an equation for the plane which passes through the points* $(1, 0, 1)$, $(0, 2, 0)$, *and* $(2, 1, 0)$.

   Solution. Let's give the three points names, say $P = (1, 0, 1)$, $Q = (0, 2, 0)$, and $R = (2, 1, 0)$. A point $S = (x, y, z)$ is in the plane if (and only if) the three vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are coplanar. As we saw in #5, this is the same as saying that $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0$.

   So now we compute some things: $\overrightarrow{PQ} = \langle -1, 2, -1 \rangle$ and $\overrightarrow{PR} = \langle 1, 1, -1 \rangle$, so $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, -2, -3 \rangle$. $\overrightarrow{PS} = \langle x - 1, y, z - 1 \rangle$, so $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0$ can be rewritten as $-1(x - 1) - 2y - 3(z - 1) = 0$, or $\boxed{x + 2y + 3z = 4}$.

   Note that it is very easy to check that this answer is correct — the three points given in the problem all satisfy this equation.

7. *True or false: If* $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, *then* $\vec{v} = \vec{w}$.

   Solution. False. There are lots of examples where this is not true. A simple one is to suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are all parallel but not equal to each other. Then, $\vec{u} \times \vec{v}$ and $\vec{u} \times \vec{w}$ are both $\vec{0}$, but $\vec{v} \neq \vec{w}$.

8. *True or false: If* $\vec{v} \times \vec{w} = \vec{0}$ *and* $\vec{v} \cdot \vec{w} = 0$, *then at least one of* $\vec{v}$ *and* $\vec{w}$ *must be* $\vec{0}$.

   Solution. True.

   Here's one way to think about it: $\vec{v} \times \vec{w} = \vec{0}$ means $\vec{v}$ and $\vec{w}$ are parallel. $\vec{v} \cdot \vec{w} = 0$ means $\vec{v}$ and $\vec{w}$ are perpendicular. The only way to have a pair of vectors that are both parallel to and perpendicular to each other is if at least one of them is the zero vector $\vec{0}$.

   If you prefer to write out equations, here's another way to think about the problem. Since $\vec{v} \times \vec{w} = \vec{0}$, $|\vec{v} \times \vec{w}| = 0$. If θ is the angle between $\vec{v}$ and $\vec{w}$, then this says that $|\vec{v}||\vec{w}| \sin\theta = 0$. On the other hand, $0 = \vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$. It's not possible for both $\sin\theta$ and $\cos\theta$ to be 0, so it must be the case that $|\vec{v}||\vec{w}| = 0$. That is, one of the vectors $\vec{v}$ and $\vec{w}$ must have length 0, and the only vector with 0 length is $\vec{0}$.